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Hyperbolic valued Multi-norms on Banach lattices

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Abstract

In this paper main concern is to study the multi-norm and a dual multi-norm with bicomplex scalars. In particular we will define $\mathbb{D}$-valued lattice multi-norm based on a Banach lattice. Also some results of hyperbolic valued multi-norms including dual multi-norms are investigated in the theory of Banach lattices.

Keywords. Bicomplex modules, hyperbolic or $\mathbb{D}$-valued norm, multi-norms, dual multi-norms, Banach lattices.

1 Introduction

Hyperbolic number system has widely been studied for various reasons, one among which is its commutative property. Infact along with the set of complex numbers, the set of hyperbolic number were found to be the only real commutative Clifford algebra. The importance of hyperbolic numbers lies in the fact that the Minkowski geometry were developed solely using this system of numbers see,[1], [9], [16], [24]. During the past several years research in this area has been to develop hyperbolic numbers as an affordable replacement for the real number system. Many papers has appeared studying hyperbolic numbers from various points of view. However in recent paper [9] studied this system of numbers as the only (natural) generalization of real numbers, into Archimedean f-algebra of dimension two. They have also generalized the fundamental properties of real numbers to this number system. The set of bicomplex numbers and the hyperbolic number system seems to have originated independently. Recently, a lots of work is being done on bicomplex numbers and bicomplex functional analysis. However later it was found that hyperbolic numbers is a subset of the set of bicomplex numbers and it plays the same role for bicomplex numbers as real numbers plays for the set of complex numbers.

Bicomplex numbers are being studied for quite a long time now. The book of G. B. Price [20] contains a comprehensive study of bicomplex numbers. A study of functional analysis with bicomplex scalars was initiated by Rochan . Recently several papers have been written on this subject , see [1], [2], [10], [14], [15], [16], [17], and references therein .

The theory of multi-normed spaces was introduced by Dales and Polyakov [5]. In this survey several properties of multi-norms and of dual multi-norms were discussed. Further details on multi-norms and dual multi-norms can be seen in [5], [6], [7], [8], [18], [19] and references therein.

The main interest of the present work lies in the study of $\mathbb{D}$-valued multi-normed space in the theory of Banach lattices.

The Paper is organized as follows. In section 2, we recall the basic notions and properties of bi-complex and hyperbolic numbers. Section 3 contains axiomatic definition of multi-normed spaces with bicomplex scalars and also discuss some immediate consequences and characterization in this section. In section 4 main topic of the paper is discussed by defining hyperbolic valued lattice multi-norm based on a Banach lattice and also defined its dual and investigated results which shows that each is the dual of the other.

2 A review of bicomplex and hyperbolic numbers

The set $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} = \{Z = z_1 + z_2 j \,|z_1, z_2 \in \mathbb{C}(i)\}$$

where i and j are commuting imaginary units such that $ij = ji$, $i^2 = j^2 =$ -1 and $\mathbb{C}(i)$ is the set of complex numbers with imaginary unit i .The set $\mathbb{BC}$ of bicomplex numbers forms a ring under usual addition and multiplication of bicomplex numbers . Moreover, $\mathbb{BC}$ is a module over itself. The product of imaginary units i and j defines a hyperbolic unit k such that $k^2 = 1$. The product of all units is commutative and satisfies

$$ij = k, ik = -j, jk = -i.$$

The set of bicomplex numbers can also be defined as

$$\mathbb{BC} = \{Z = x_0 + ix_1 + jx_2 + ijx_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\}.$$

If we put $z_1 = x$ and $z_2 = iy$ with $x, y \in \mathbb{R}$, then the set of hyperbolic numbers denoted by $\mathbb{D}$ is a ring of all numbers of the form $Z = x + yk$, with k satisfying $k^2 = 1$, i.e.,

$$\mathbb{D} = \left\{x + yk \,|x, y \in \mathbb{R}, k^2 = 1, k \notin \mathbb{R}\right\}.$$

The set $\left\{x + yij \,|x, y \in \mathbb{R}, i^2 = j^2 = -1\right\}$ is a subset of the set of bicomplex numbers which is isomorphic to $\mathbb{D}$ as a real algebra.
Since $\mathbb{BC}$ contains two imaginary units which squares to -1 and one hyperbolic unit which squares to 1 ,the following three conjugations are considered for bicomplex numbers. With
$Z = z_1 + z_2 j$, we define
(i) $\overline{Z} = \overline{z_1} + \overline{z_2} j$ (the bar-conjugation)
(ii) $Z^\dagger = z_1 - z_2 j$ (the†-conjugation)
(iii) $Z^* = (\overline{Z})^\dagger = \overline{(Z^\dagger)} = \overline{z_1} - \overline{z_2} j$ (the $*$-conjugation),
where $\overline{z_1}$, $\overline{z_2}$ denote the usual complex conjugates of z_1, $z_2 \in \mathbb{C}(i)$.
With each kind of conjugation, one can define a specific bicomplex modulus as;
(i) $|Z|_i^2 = ZZ^\dagger = z_1^2 + z_2^2, \in \mathbb{C}(i)$
$(ii)|Z|_j^2 = Z\overline{Z} = \left(|z_1|^2 - |z_2|^2\right) + 2Re\left(z_1\overline{z_2}\right) j, \in \mathbb{C}(j)$
$(iii)|Z|_k^2 = ZZ^* = \left(|z_1|^2 + |z_2|^2\right) - 2Im\left(z_1\overline{z_2}\right) k, \in \mathbb{D}$
Since none of the moduli above is real valued, we can consider also the Euclidean norm on $\mathbb{BC}$, that is, for any $Z = x_0 + ix_1 + jx_2 + ijx_3 = z_1 + z_2 j \in \mathbb{BC}$, define

$$|Z| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{Re(|Z|_k^2)}.$$

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And it can be easily check that for any $Z, W \in \mathbb{BC}$, we have $|Z.W| \leq \sqrt{2}\,|Z|\,|W|$.
As we know that for any bicomplex number $Z = z_1 + z_2 j$, we have

$$Z\frac{Z^{\dagger}}{|Z|_i^2} = 1,$$

then the inverse of Z is given by

$$Z^{-1} = \frac{Z^{\dagger}}{|Z|_i^2}.$$

If both z_1 and z_2 are non zero but the sum $z_1^2 + z_2^2 = 0$,then the corresponding bicomplex number Z is a zero divisor . From this we find the set $\mathcal{NC}$ of zero divisors of $\mathbb{BC}$ called the null cone is given by

$$\mathcal{NC} = \left\{Z = z_1 + z_2 j; Z \neq 0, Z.Z^{\dagger} = z_1^2 + z_2^2 = 0\right\}.$$

This introduces the two very special zero divisors defined as

$$e_1 = \tfrac{1+ij}{2} \text{ and } e_2 = \tfrac{1-ij}{2}.$$

Infact, e_1 and e_2 are hyperbolic numbers. It is easy to see e_1 and e_2 are zero divisors and are mutually complementary idempotent elements, such that

$$(e_1)^2 = e_1, \quad (e_2)^2 = e_2, \quad e_1 + e_2 = 1$$

$$(e_1)^* = e_1, \quad (e_2)^* = e_2, \quad e_1 e_2 = 0.$$

The two principal ideals in $\mathbb{BC}$ are generated by e_1 and e_2 are denoted by $\mathbb{BC}_{e_1}$ and $\mathbb{BC}_{e_2}$ where,

$$\mathbb{BC}_{e_1} = e_1\mathbb{BC}\ , \quad \mathbb{BC}_{e_2} = e_2\mathbb{BC}$$

with

$$\mathbb{BC}_{e_1} \cap \mathbb{BC}_{e_2} = \{0\}$$

and

$$\mathbb{BC} = e_1\mathbb{BC} + e_2\mathbb{BC} \qquad (1.1).$$

The representation (1.1) is called the idempotent decomposition of $\mathbb{BC}$.
The two hyperbolic numbers e_1 and e_2 has simplified the bicomplex algebra by allowing us to a unique idempotent representation of $\mathbb{BC}$ in the following form ;
every bicomplex number

$$Z = z_1 + z_2 j \in \mathbb{C}^2(i)$$

can be written as

$$Z = \beta_1 e_1 + \beta_2 e_2,$$

where

$$\beta_1 = z_1 - iz_2 \quad and \quad \beta_2 = z_1 + iz_2 \in \mathbb{C}(i). \qquad (1.2)$$

The hyperbolic-valued or $\mathbb{D}$-valued norm $|Z|_k$ of a bicomplex number $Z = \beta_1 e_1 + \beta_2 e_2$ is defined as

$$|Z|_k = |\beta_1| e_1 + |\beta_2| e_2,$$

where $|\beta_1|$ and $|\beta_2|$ are the usual modulus of complex numbers β_1 and β_2. Further $|Z.W|_k = |Z|_k\,|W|_k$ and Euclidean norm and hyperbolic norm of a bicomplex number is related by $||Z|_k| = |Z|$.

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For more details on Euclidean norm and hyperbolic norm one can refer to [1].
We can also decompose the set $\mathbb{D}$ of hyperbolic numbers as

$$\mathbb{D} = \mathbb{D}e_1 + \mathbb{D}e_2 \qquad (1.3).$$

The representation (1.3) is called the idempotent decomposition of $\mathbb{D}$. Thus the idempotent representation of a hyperbolic number $\alpha = x + yk$ is

$$\alpha = \alpha_1 e_1 + \alpha_2 e_2$$

with

$$\alpha_1 = x + y \quad and \quad \alpha_2 = x - y \in \mathbb{R}.$$

We say that α is positive if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. Thus we have the set

$$\mathbb{D}^+ = \{\alpha_1 e_1 + \alpha_2 e_2; \alpha_1, \alpha_2 \geq 0\},$$

which shows that positive hyperbolic numbers are those whose both idempotent components are non negative .

In [1] and [9] a partial order relation , supremum and infimum in $\mathbb{D}$ are defined as follows ;
Given for $x, y \in \mathbb{D}$, define a relation $\preceq$ on $\mathbb{D}$ by $x \preceq y$ whenever $y - x \in \mathbb{D}^+$. This relation is reflexive,transitive and antisymmetric and therefore it defines a partial order on $\mathbb{D}$. If we take $a, b \in \mathbb{R}$,then $a \preceq b$ if and only if $a \leq b$, and so $a \preceq b$ is an extension of the total order $\leq$ on $\mathbb{R}$.
The following are the properties of order $\preceq$ which will be useful in subsequent results. Let $x, y, z, w \in \mathbb{D}$.

(1) If $x \preceq y$ and $z \in \mathbb{D}^+$, then $zx \preceq zy$

(2) If $x \preceq y$ and $z \preceq w$, then $x + z \preceq y + w$.

(3) If $x \preceq y$, then $-y \preceq -x$.

Let $A \subset \mathbb{D}$. If there exists $M \in \mathbb{D}^+$ such that $|x|_k \preceq M \ \forall x \in A$, we say that A is a $\mathbb{D}$-bounded set. If $A \subset \mathbb{D}$ is a $\mathbb{D}$-bounded from above , then the $\mathbb{D}$-supremum of A is defined as

$$\sup_{\mathbb{D}} A = \sup A_1 e_1 + \sup A_2 e_2,$$

where

$$A_1 = \{x \in \mathbb{R} | \exists \quad y \in \mathbb{R}, xe_1 + ye_2 \in A\},$$

$$A_2 = \{y \in \mathbb{R} | \exists \quad x \in \mathbb{R}, xe_1 + ye_2 \in A\}.$$

Similarly, $\mathbb{D}$-infimum of a $\mathbb{D}$-bounded below set A is defined as

$$\inf_{\mathbb{D}} A = \inf A_1 e_1 + \inf A_2 e_2,$$

where A_1 and A_2 are as defined above.

The algebra of hyperbolic numbers is endowed with a partial order structure. Under a well defined order the hyperbolic numbers is the only generalization of real numbers into Archimedean **f**-algebra of dimension two. As a consequence fundamental order properties , including Dedekind completness, can be obtained. For more details see, [9], it is also proved that $\mathbb{D}$ is a Banach lattice.
Since X_{e_1} and X_{e_2} are $\mathbb{R}$-, $\mathbb{C}(i)$- and $\mathbb{C}(j)$- linear spaces as well as $\mathbb{BC}$-modules, we have that

$$X = X_{e_1} \oplus X_{e_2},$$

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where the direct sum $\oplus$ can be understood in the sense of $\mathbb{R}$-, $\mathbb{C}(i)$- or $\mathbb{C}(j)$- linear spaces, as well as $\mathbb{BC}$-modules.

Definition1.1. Let X be a $\mathbb{BC}$-module. Then X can be written as

$$X = e_1 X_1 + e_2 X_2,$$

where $X_1 = e_1 X$ and $X_2 = e_2 X$ are two $\mathbb{C}(i)$-linear spaces. Thus each $x \in X$ can be uniquely written as $x = xe_1 + xe_2 = x_1 e_1 + x_2 e_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. A real-valued norm on $\mathbb{BC}$-module X is defined as

$$\|x\| = \frac{1}{\sqrt{2}}\sqrt{\|x_1\|_1^2 + \|x_2\|_2^2}$$

for any $x \in X$, where $\|.\|_1$ and $\|.\|_2$ are real-valued norm on X_1 and X_2. However, for any scalar $\lambda \in \mathbb{BC}$ and $x \in X$, norm satisfies the inequality

$$\|\lambda x\| \leq \sqrt{2}\,|\lambda|\,\|x\|\,.$$

The $\mathbb{BC}$-module X can be endowed canonically with the hyperbolic-valued, or $\mathbb{D}$-valued, norm denoted by $\|.\|_{\mathbb{D}}$ as follows :

$$\|x\|_{\mathbb{D}} = \|e_1 x_1 + e_2 x_2\|_{\mathbb{D}} = \|x_1\|_1\, e_1 + \|x_2\|_2\, e_2 \qquad (1.4)$$

such that for any $\lambda \in \mathbb{BC}$,and $x, y \in X$, we have

$$\|\lambda x\|_{\mathbb{D}} = |\lambda|_k\, \|x\|_{\mathbb{D}} \text{ and } \|x + y\|_{\mathbb{D}} \preceq \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}} \qquad (1.5)$$

where $\preceq$ is a partial order relation on $\mathbb{D}$.

The comparison of real valued norm $\|x\|$ and $\mathbb{D}$ valued norm $\|x\|_{\mathbb{D}}$ of $x \in X$ gives

$$|\|x\|_{\mathbb{D}}| = \|x\|.$$

For details on real-valued norm and hyperbolic-valued norm see, [1,sections 4.1, 4.2].

3 Hyperbolic-valued multi-norms

In this section our aim is to extend the multinorms in the ring of bicomplex numbers from its complex version.

Let $(X, \|.\|_{\mathbb{D}})$ be a $\mathbb{D}$-valued normed space, and for $m \in \mathbb{N}$, X^m denotes the linear space $X \oplus ... \oplus X$ consisting of m-tuples $(x_{1,1}\, e_1 + x_{1,2}\, e_2, ..., x_{m,1}\, e_1 + x_{m,2}\, e_2)$, where $x_{1,1}\, e_1 + x_{1,2}\, e_2, ..., x_{m,1}\, e_1 + x_{m,2}\, e_2 \in X$. The linear operations on X^m are defined coordinate-wise. We write $\mathbb{N}$ for the set of natural numbers ; and for $n \in \mathbb{N}$, $\mathbb{N}_m$ denotes the set $\{1, 2, ...m\}$.

We begin with the definition of a hyperbolic(or $\mathbb{D}$-valued) muti-norm.

Definition 3.1. Let $(X, \|.\|_{\mathbb{D}})$ be a $\mathbb{D}$-valued normed space and take $n \in \mathbb{N}$. A hyperbolic (or $\mathbb{D}$-valued) multi-norm of level n on $\{X^m : m \in \mathbb{N}_n\}$ is a sequence $(\|.\|_{\mathbb{D},m}) = (\|.\|_{\mathbb{D},m} : m \in \mathbb{N}_n)$ such that $\|.\|_{\mathbb{D},m}$ is a norm on X^m for each $m \in \mathbb{N}_n$, with $\|.\|_{\mathbb{D},1} = \|.\|_{\mathbb{D}}$ on $X = X^1$, and such that the following axioms hold for each $m \in \mathbb{N}_n$ with $m \geq 2$ and $x = (x_1, ..., x_m) \in X^m$;

(M1) $\left\|\left(x_{\sigma(1)}, ..., x_{\sigma(m)}\right)\right\|_{\mathbb{D},m} = \|(x_1, ..., x_m)\|_{\mathbb{D},m}$, for each permutation σ of $\mathbb{N}_n$

(M2) $\|(\mu_1 x_1, ..., \mu_l x_m)\|_{\mathbb{D},m} \preceq (max_{i\in\mathbb{N}_m} |\mu_i|_k)\, \|(x_1, ..., x_m)\|_{\mathbb{D},m}$, for each $\mu_1, .., \mu_m \in \mathbb{BC}$

(M3) $\|(x_1,....,x_m,0)\|_{\mathbb{D},m+1} = \|(x_1,....,x_l)\|_{\mathbb{D},m}$
(M4) $\|(x_1,....,x_m,x_m)\|_{\mathbb{D},m+1} = \|(x_1,....,x_m)\|_{\mathbb{D},m}$
In this case $((X^m, \|.\|_{\mathbb{D},m}) : m \in \mathbb{N}_n)$ is a $\mathbb{D}$-valued multi-normed space of level n. We can also say that $(\|.\|_{\mathbb{D},m} : m \in \mathbb{N})$ is a $\mathbb{D}$-valued multi-norm based on X.
Since X is of the form $X = X_1e_1+X_2e_2$ and X^n contains the n-tuples $(x_1....,x_n)$ where $x_1,....,x_n \in X$. Thus each $x = (x_1,....,x_n) \in X^n$, can be written as $x_n = x_{n,1}e_1 + x_{n,2}e_2$.
Also note that Axioms (M1) and (M4) together say precisely that , for each $n \in \mathbb{N}$, the value of $\|x_1,....,x_n\|_{\mathbb{D},n}$ depends on only the set $\{x_1,....,x_n\}$. Next we will define $\mathbb{D}$-valued dual multi-norm, which follows from [5].

Definition 3.2. Let $(X, \|.\|_{\mathbb{D}})$ be a $\mathbb{D}$-valued normed space and let $(\|.\|_{\mathbb{D},m}) = (\|.\|_{\mathbb{D},m} : m \in \mathbb{N})$ be a sequence such that $\|.\|_{\mathbb{D},m}$ is a norm on X^m for each $m \in \mathbb{N}$, such that $\|x\|_{\mathbb{D},1}=\|x\|_{\mathbb{D}}$ for each $x \in X$. Then the sequence $(\|.\|_{\mathbb{D},m} : m \in \mathbb{N})$ is a $\mathbb{D}$-valued dual multi-norm on $\{X^m : m \in \mathbb{N}\}$ if the Axioms (M1),(M2),(M3) and the modified form of axiom (M4) which is (D4), are satisfied for each $m \in \mathbb{N}$ with $m \geq 2$ and $x = (x_1,.....,x_m) \in X^m$,
(D4) $\|(x_1,.....,x_m,x_m)\|_{\mathbb{D},m+1} = \|(x_1,....,x_{m-1},2x_m)\|_{\mathbb{D},m}$
In this case we say that $((X^m, \|.\|_{\mathbb{D},m}) : m \in \mathbb{N})$ is a $\mathbb{D}$-valued dual multi normed space.
In the above situation $(\|.\|_{\mathbb{D},m} : m \in \mathbb{N})$ is known as hyperbolic (or $\mathbb{D}$-valued) dual multi-norm based on X.
Suppose that $((X^m, \|.\|_{\mathbb{D},m}) : m \in \mathbb{N})$ is a $\mathbb{D}$-valued multi-normed space and take $m \in \mathbb{N}$. It is easy to show that
(1) $\|((xe_1+xe_2),...,(xe_1+xe_2))\|_{\mathbb{D},m} = \|xe_1+xe_2\|_{\mathbb{D}}$
(2)

$$\max_{l\in\mathbb{N}_m} \|x_{l,1}e_1 + x_{l,2}e_2\|_{\mathbb{D}} \preceq \|((x_{1,1}e_1+x_{1,2}e_2),...,(x_{m,1}e_1+x_{m,2}e_2))\|_{\mathbb{D},m}$$

$$\preceq \sum_{l=1}^{m} \|x_{l,1}e_1+x_{l,2}e_2\|_{\mathbb{D}} \preceq m \max_{l\in\mathbb{N}_m} \|x_{l,1}e_1+x_{l,2}e_2\|_{\mathbb{D}}.$$

It follows from (2) that if $(X, \|.\|_{\mathbb{D}})$ is a Banach space then, $(X^m, \|.\|_{\mathbb{D},m})$ is a Banach space for each $m \in \mathbb{N}$; in this case $((X^m, \|.\|_{\mathbb{D},m}) : n \in \mathbb{N})$ is a $\mathbb{D}$-valued multi-Banach space or $\mathbb{D}$-valued dual multi-Banach space respectively.

Theorem 3.3. Let X^n be a bicomplex multi-normed space.Then $X^n = X_{n,1}e_1 + X_{n,2}e_2$ is a bicomplex multi-Banach space if and only if $X_{n,1}$ and $X_{n,2}$ are complex multi-Banach spaces.

Proof. Firstly suppose that $X^n = X_{n,1}e_1 + X_{n,2}e_2$ is a bicomplex multi-Banach space. Let $\{x^m_{n,i}\}_{m=0}^{\infty}$ be a Cauchy sequence in $X_{n,i}$ for $i = 1,2$, where $\forall m \in \mathbb{N}$, $x^m_{n,i} = e_ix^m_n$. Thus $\{x^m_n\}_{m=0}^{\infty}$ is a Cauchy sequence in X^n. But X^n is a Banach space, so for given $\epsilon \succ 0$ and $x_n \in X^n$,there exists $p \in \mathbb{N}$ such that $\|x^m_n - x_n\|_{\mathbb{D}} \prec \epsilon; \forall m \geq p$.
Now $\|e_ix^m_n - e_ix_n\|_{\mathbb{D}} = \|e_i(x^m_n - x_n)\|_{\mathbb{D}} = |e_i|_k \|x^m_n - x_n\|_{\mathbb{D}} = \frac{1}{\sqrt{2}}\|x^m_n - x_n\|_{\mathbb{D}} \prec \epsilon, \forall m \geq p$.
This means that $e_ix^m_n \to e_ix_n$, for $i = 1,2$.
Hence $X_{n,i}$, for $i = 1,2$ is a complex banach space.
Conversely suppose that $\{x^m_n = x^m_ne_1 + x^m_ne_2\}_{m=0}^{\infty}$ be a Cauchy sequence in X^n,then $\{x^m_{n,i}\}_{m=0}^{\infty} = \{e_ix^m_n\}_{m=0}^{\infty}$ is a Cauchy sequence in$X_{n,i}$ for $i = 1,2$. By using completeness of $X_{n,i}$,it is easy to show that X^n is complete. Hence X^n is a bicomplex Banach space.
Many properties of $\mathbb{D}$-valued multi-norms and of $\mathbb{D}$-valued dual multi-norms are described in [25] .
Duality It will be convenient to designate elements of the dual space $X^\star$ of X by $x^\star = x^\star e_1 + x^\star e_2$ and to write $\langle x, x^\star\rangle$ in place of $x^\star(x)$. This notation is well adapted to the symmetry (or duality) that exists between the action of $X^\star$ on X on the one hand and the action of X on $X^\star$.
Let $\|.\|_{\mathbb{D},n}$ be a $\mathbb{D}$-valued norm on $\mathbb{BC}$-module X^n. Then $\|.\|^\star_{\mathbb{D},n}$ is the $\mathbb{D}$-valued dual norm on

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$(X^n)^\star$ i.e., the dual space of a $\mathbb{D}$-valued normed space $(X, \|.\|_{\mathbb{D}})$ is denoted by $X^\star$ and the action of $x^\star e_1 + x^\star e_2 \in X^\star$ on $xe_1 + xe_2 \in X$ gives the number $\langle xe_1 + xe_2, x^\star e_1 + x^\star e_2 \rangle$.
The following results establish duality and which are proved in [25] .

Theorem 3.4. Let $((X^m, \|.\|_{\mathbb{D},m}); m \in \mathbb{N})$ be a $\mathbb{D}$-valued multi-normed space. Then

$$(((X^\star)^m, \|.\|^\star_{\mathbb{D},m}) : m \in \mathbb{N})$$

is a $\mathbb{D}$-valued dual multi-Banach space.

Theorem 3.5. Let $((Y^m, \|.\|_{\mathbb{D},m}); m \in \mathbb{N})$ be a $\mathbb{D}$-valued dual multi-normed space. Then

$$(((Y^\star)^m, \|.\|^\star_{\mathbb{D},m}) : m \in \mathbb{N})$$

is a $\mathbb{D}$-valued multi-Banach space.

4 $\mathbb{D}$-valued Multi-norms on Banach lattices

Definition 4.1. A norm $\|.\|_{\mathbb{D}}$ on Riesz space $\mathbb{D}$ is called Riesz norm or (lattice norm) if $\|u\|_{\mathbb{D}} \preceq \|v\|_{\mathbb{D}}$ whenever $|u|_k \leq |v|_k$ in $\mathbb{D}$. A Reisz space equipped by a Riesz norm is called normed Riesz space. A complete normed Riesz space is called Banach lattice.
Banach lattice structure for hyperbolic numbers is well-defined in [9].
The ordering on Riesz space $\mathbb{D}$ is determined by $\mathbb{D}^+$. For $x, y \in \mathbb{D}$ the operations
$(xe_1 + xe_2, ye_1 + ye_2) \mapsto xe_1 + xe_2 \vee ye_1 + ye_2$ and $(xe_1 + xe_2, ye_1 + ye_2) \mapsto xe_1 + xe_2 \wedge ye_1 + ye_2$ are the lattice operations, and are defined pointwise, i.e.,
$(xe_1 + xe_2 \vee ye_1 + ye_2) = \max\{xe_1 + xe_2, ye_1 + ye_2\}$
$(xe_1 + xe_2 \wedge ye_1 + ye_2) = \min\{xe_1 + xe_2, ye_1 + ye_2\}$
The supremum and infimum of a non-empty subset L of a lattice $\mathbb{D}$ are denoted by $\bigvee L$ and $\bigwedge L$ respectively.

Definition 4.2. Let $\mathbb{D}$ be a Banach lattice. Then for $z \in \mathbb{D}$, we have $z = x + yk$, where $x, y \in \mathbb{R}$, the modulus $|z|_k \in \mathbb{D}^+$ of z is defined by

$$|z|_k = \bigvee \{(xcos\theta + ysin\theta)e_1 + (xcos\theta + ysin\theta)e_2 : 0 \leq \theta \leq 2\pi\} \qquad (1.6)$$

We see that for $\alpha \in \mathbb{BC}$ and $z, w \in \mathbb{D}$, we have following properties:
(1)$|z|_k = 0$ if and only if $z = 0$;
(2)$|\alpha z|_k = |\alpha|\ |z|_k$;
(3)$|z + w|_k \preceq |z|_k + |w|_k$.

Definition 4.3. Let $\mathbb{D}$ be a Banach lattice, with dual space $\mathbb{D}^\star$. Then $\mathbb{D}^\star$ is ordered by the requirement that $u \in \mathbb{D}^\star$ belongs to $(\mathbb{D}^\star)^+$ if and only if for $x \in \mathbb{D}^+$, $\langle x, u \rangle \succeq 0$, and then $\mathbb{D}^\star$ becomes a Banach lattice with respect to the following definitions of $u \vee v$ and $u \wedge v$ for $u, v \in \mathbb{D}^\star$. Infact $u \vee v$ and $u \wedge v$ are defined for $x \in \mathbb{D}^+$ by

$$\begin{cases} \langle xe_1 + xe_2, (ue_1 + ue_2) \vee (ve_1 + ve_2) = sup\{\langle ye_1 + ye_2, ue_1 + ue_2\rangle + \langle ze_1 + ze_2, ve_1 + ve_2\rangle : u, v \in \mathbb{D}^+, y + z = x\} \\ \langle xe_1 + xe_2, (ue_1 + ue_2) \wedge (ve_1 + ve_2)\rangle = inf\{\langle ye_1 + ye_2, ue_1 + ue_2\rangle + \langle ze_1 + ze_2, ve_1 + ve_2\rangle : u, v \in \mathbb{D}^+, y + z = x\} \end{cases} \qquad (1.7)$$

and then $u \vee v$ and $u \wedge v$ are extended to $\mathbb{D}^\star$. The dual of a Banach lattice $\mathbb{D}$ is also a Banach lattice; and this is the dual Banach lattice of $\mathbb{D}$.
Let $\mathbb{D}$ be a Banach lattice, and take $x \in \mathbb{D}^+$ and let $u \in \mathbb{D}^\star$. Then we have

$$\langle xe_1 + xe_2, u^+e_1 + u^+e_2\rangle = sup\{\langle ye_1 + ye_2, ue_1 + ue\rangle : 0 \preceq y \preceq x\}.$$

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Also if $\mathbb{D}$ is a Banach lattice, then for $z \in \mathbb{D}$ and $u \in \mathbb{D}^{\star}$ we have

$$|\langle ze_1 + ze_2, ue_1 + ue\rangle|_k \preceq \langle |ze_1 + ze_2|_k, |ue_1 + ue_2|_k\rangle . \quad (1.8)$$

Proposition 4.4. Let $\mathbb{D}$ be a Banach lattice, and take $x \in \mathbb{D}^{+}$ and let $u \in \mathbb{D}^{\star}$ and $\epsilon \succ 0$. Then there exists $z \in \mathbb{D}$ such that

$$|ze_1 + ze_2|_k \preceq xe_1 + xe_2 \quad \text{and } |\langle ze_1 + ze_2, ue_1 + ue\rangle|_k \succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon . \quad (1.9)$$

Proof : Since $u \in \mathbb{D}^{\star}$, therefore for $\alpha, \beta \in \mathbb{R}$, we have $u = \alpha + \beta k$ and by definition 4.2., we have

$$|ue_1 + ue_2|_k = \bigvee \{(\alpha cos\theta + \beta sin\theta)e_1 + (\alpha cos\theta + \beta sin\theta)e_2 : 0 \leq \theta \leq 2\pi\} ,$$

and so there exists $\theta_1, ..., _n \in [0, 2\pi]$ such that,

$$\langle xe_1 + xe_2, ((\alpha cos\theta_1 + \beta sin\theta_1)e_1 + (\alpha cos\theta_2 + \beta sin\theta_2)e_2) \vee ... \vee (\alpha cos\theta_n + \beta sin\theta_n)e_1 + ((\alpha cos\theta_n + \beta sin\theta_n)e_2)\rangle$$
$$\succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon.$$

By extending the definition in (1.7), there exists $y_1, ..., y_n \in \mathbb{D}^{+}$ such that $y_1 + ... + y_n = x$ and

$$\langle ye_1 + ye_2, ((\alpha cos\theta_1 + \beta sin\theta_1)e_1 + (\alpha cos\theta_2 + \beta sin\theta_2)e_2)\rangle + ... + \langle y_n e_1 + y_n e_2, ((\alpha cos\theta_1 + \beta sin\theta_1)e_1 + (\alpha cos\theta_1 + \beta sin\theta_2)e_2)\rangle$$
$$\succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon.$$

That is,

$$\sum_{l=i}^{n} \langle (cos\theta_l e_1 + cos\theta_l e_2)(y_l e_1 + y_l e_2), \alpha e_1 + \alpha e_2\rangle + \sum_{l=i}^{n} \langle (sin\theta_l e_1 + sine\theta_l e_2)(y_l e_1 + y_l e_2), \beta e_1 + \beta e_2\rangle$$
$$\succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon. \quad (1.10)$$

Let $t \in \mathbb{D}$ be such that

$$t = \sum_{l=i}^{n} ((cos\theta_l e_1 + cos\theta_l e_2) - (sin\theta_l e_1 + sine\theta_l e_2))(y_l e_1 + y_l e_2).$$

From equation (1.10), we have $\langle te_1 + te_2, ue_1 + ue_2\rangle \succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon$, and so

$$|\langle te_1 + te_2, ue_1 + ue_2\rangle|_k \succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon.$$

For each $\theta \in [0, 2\pi]$, we have

$$\sum_{l=i}^{n} ((cos\theta e_1 + cos\theta e_2)(cos\theta_l e_1 + cos\theta_l e_2) - (sin\theta e_1 + sin\theta e_2)(sin\theta_l e_1 + sin\theta_l e_2)(y_l e_1 + y_l e_2))$$
$$= \sum_{l=i}^{n} cos((\theta e_1 + \theta e_2) + (\theta_l e_1 + \theta_l e_2))(y_l e_1 + y_l e_2),$$

and hence

$$|te_1 + te_2|_k = sup \left\{ \sum_{l=i}^{n} cos((\theta e_1 + \theta e_2) + (\theta_l e_1 + \theta_l e_2))(y_l e_1 + y_l e_2) : 0 \preceq \theta \preceq 2\pi \right\}$$
$$\preceq \sum_{l=i}^{n} (y_l e_1 + y_l e_2) = xe_1 + xe_2.$$

Now, set $(ze_1 + ze_2) = (\xi e_1 + \xi e_2)\ (te_1 + te_2)$, where $(\xi e_1 + \xi e_2) \in \mathbb{D}_{\mathbb{BC}}$ is chosen such that $(\xi e_1 + \xi e_2)\ \langle te_1 + te_2, ue_1 + ue_2\rangle = |\langle te_1 + te_2, ue_1 + ue_2\rangle|_k$.
Then

$$|ze_1 + ze_2|_k = |te_1 + te_2|_k \preceq (xe_1 + xe_2)$$

and

$$|\langle ze_1 + ze_2, ue_1 + ue\rangle|_k \succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon.$$

Definition 4.5. Let $(X, \|.\|_{\mathbb{D}})$ be a $\mathbb{D}$-valued Banach lattice. For each $(x_{1,1}e_1 + x_{1,2}e_2, ..., x_{n,1}e_1 + x_{n,2}e_2) \in X$, set

$$\|(x_{1,1}e_1 + x_{1,2}e_2, ..., x_{n,1}e_1 + x_{n,2}e_2)\|^L_{\mathbb{D},n} = \||x_{1,1}e_1 + x_{1,2}e_2|_k \vee ..... \vee |x_{n,1}e_1 + x_{n,2}e_2|_k\|_{\mathbb{D}}$$

and

$$\|(x_{1,1}e_1 + x_{1,2}e_2, ..., x_{n,1}e_1 + x_{n,2}e_2)\|^{DL}_{\mathbb{D},n} = \||x_{1,1}e_1 + x_{1,2}e_2|_k + ... + |x_{n,1}e_1 + x_{n,2}e_2|_k\|_{\mathbb{D}}.$$

Then $(X^n, \|.\|^L_{\mathbb{D},n})$ is a $\mathbb{D}$-valued multi-Banach space. It is the $\mathbb{D}$-valued Banach lattice multi-norm. Also $(X^n, \|.\|^{DL}_{\mathbb{D},n})$ is a $\mathbb{D}$-valued dual multi-Banach space. It is the $\mathbb{D}$-valued dual Banach lattice multi-norm.

And each is the dual of the other.

Theorem 4.6. Let $(X, \|.\|_{\mathbb{D}})$ be a Banach lattice. Then the dual of $\mathbb{D}$-valued lattice multi-norm on $\{X^n : n \in \mathbb{N}\}$ is the dual $\mathbb{D}$-valued lattice multi-norm on $\{(X^\star)^n : n \in \mathbb{N}\}$.

Proof : Let $(\|.\|^L_{\mathbb{D},n})$ be the $\mathbb{D}$-valued lattice multi-norm on the family $\{X^n : n \in \mathbb{N}\}$. Then $\|.\|^\star_{\mathbb{D},n}$ is the dual $\mathbb{D}$-valued norm to $\|.\|^L_{\mathbb{D},n}$, for $n \in \mathbb{N}$. Now, for $(u_{1,1}e_1 + u_{1,2}e_2, ...u_{n,1}e_1 + u_{n,2}e_2 \in X^\star)$ we have to prove that

$\|(u_{1,1}e_1 + u_{1,2}e_2, ...u_{n,1}e_1 + u_{n,2}e_2)\|^\star_{\mathbb{D},n} = \||u_{1,1}e_1 + u_{1,2}e_2|_k + ... + |u_{n,1}e_1 + u_{n,2}e_2|_k\|_{\mathbb{D}}$

Let $ue_1 + ue_2 = |u_{1,1}e_1 + u_{1,2}e_2|_k + ... + |u_{n,1}e_1 + u_{n,2}e_2|_k$.

Suppose that $(x_{1,1}e_1 + x_{1,2}e_2, ...x_{n,1}e_1 + x_{n,2}e_2) \in X$ with

$\|(x_{1,1}e_1 + x_{1,2}e_2, ...x_{n,1}e_1 + x_{n,2}e_2\|^L_{\mathbb{D},n} \preceq 1$,and let

$$xe_1 + xe_2 = |x_{1,1}e_1 + x_{1,2}e_2|_k \vee .... \vee |x_{n,1}e_1 + x_{n,2}e_2|_k,$$

so that$\|xe_1 + xe_2\|_{\mathbb{D}} \preceq 1$.

Using equation (1.8),we have

$$|\langle(x_{1,1}e_1 + x_{1,2}e_2, ...x_{n,1}e_1 + x_{n,2}e_2), (u_{1,1}e_1 + u_{1,2}e_2, ...u_{n,1}e_1 + u_{n,2}e_2)\rangle|_k \preceq \sum_{l=1}^{n} |\langle x_{l,1}e_1 + x_{l,2}e_2, u_{l,1}e_1 + u_{l,2}e_2\rangle|_k$$

$$\preceq \sum_{l=1}^{n} \langle |x_{l,1}e_1 + x_{l,2}e_2, u_{l,1}e_1 + u_{l,2}e_2|_k, |u_{1,1}e_1 + u_{1,2}e_2, ...u_{n,1}e_1 + u_{n,2}e_2|_k\rangle \preceq \langle xe_1 + xe_2, ue_1 + ue_2\rangle$$

which shows that

$\|(u_{1,1}e_1 + u_{1,2}e_2, ...u_{n,1}e_1 + u_{n,2}e_2)\|^\star_{\mathbb{D},n} \preceq \|ue_1 + ue_2\|_{\mathbb{D}}$.

Given $\epsilon \succ 0$,there exits $x \in \mathbb{D}^+$ with $\|xe_1 + xe_2\|_{\mathbb{D}} = 1$ and

$$\langle xe_1 + xe_2, ue_1 + ue_2\rangle \succ \|ue_1 + ue_2\|_{\mathbb{D}} - \epsilon.$$

Now using Proposition 4.4.we see that for each $l \in \mathbb{N}_n$, there exists $y_{l,1}e_1 + y_{l,2}e_2 \in \mathbb{D}$ with $|y_{l,1}e_1 + y_{l,2}e_2|_k \preceq xe_1 + xe_2$ and

$$\langle y_{l,1}e_1 + y_{l,2}e_2, ue_1 + ue_2\rangle \succ \langle xe_1 + xe_2, |ue_1 + ue_2|_k\rangle - \epsilon.$$

Also, $|y_{1,1}e_1 + y_{1,2}e_2|_k \vee .... \vee |y_{n,1}e_1 + y_{n,2}e_2|_k$, and so

$$\|(y_{1,1}e_1 + y_{1,2}e_2, ..., y_{n,1}e_1 + y_{n,2}e_2)\|^L_{\mathbb{D},n} = \||y_{1,1}e_1 + y_{1,2}e_2|_k \vee .... \vee |y_{n,1}e_1 + y_{n,2}e_2|_k\|_{\mathbb{D}} \preceq \|xe_1 + xe_2\|_{\mathbb{D}} \preceq 1.$$

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Also,

$$|\langle (y_{1,1}e_1+y_{1,2}e_2,..,y_{n,1}e_1+y_{n,2}e_2),(u_{1,1}e_1+u_{1,2}e_2,..,u_{n,1}e_1+u_{n,2}e_2)\rangle|_k = \left|\sum_{l=1}^{n}\langle y_{l,1}e_1+y_{l,2}e_2,u_{l,1}e_1+u_{l,2}e_2\rangle\right|_k$$
$$\succeq \sum_{l=1}^{n}\langle xe_1+xe_2,|u_{l,1}e_1+u_{l,2}e_2|_k\rangle - n\epsilon$$
$$= \langle xe_1+xe_2,|ue_1+ue_2|_k\rangle - n\epsilon$$
$$\succ \|ue_1+ue_2\|_{\mathbb{D}} - (n+1)\epsilon,$$

and so $\|(u_{1,1}e_1+u_{1,2}e_2,..,u_{n,1}e_1+u_{n,2}e_2)\|^{\star}_{\mathbb{D},n} \succeq \|ue_1+ue_2\|_{\mathbb{D}}-(n+1)\epsilon$.This holds true for each $\epsilon \succ 0$, and so $\|(u_{1,1}e_1+u_{1,2}e_2,..,u_{n,1}e_1+u_{n,2}e_2)\|^{\star}_{\mathbb{D},n} \succeq \|ue_1+ue_2\|_{\mathbb{D}}$.
This proves the result.

Theorem 4.7. Let $(X,\|.\|_{\mathbb{D}})$ be a Banach lattice. Then the dual of the dual $\mathbb{D}$-valued lattice multi-norm on $\{X^n : n\in\mathbb{N}\}$ is the $\mathbb{D}$-valued lattice multi-norm on $\{(X^{\star})^n : n\in\mathbb{N}\}$.

Proof : The proof for this is similar to 4.6.

Example 4.8. Let $\Omega_{\mathbb{D}}$ be a $\mathbb{D}$-valued measure space,and take $p\succeq 1$. Then for each $q\succeq p$, the standard q - hyperbolic valued multi-norm based on $L^p(\Omega_{\mathbb{D}})$ is the hyperbolic-valued ($\mathbb{D}$-valued) multi-norm $(\|.\|^{[q]}_{\mathbb{D},n} : n\in\mathbb{N})$.

We denote the hyperbolic norm as $\|(f_{1,1}e_1+f_{1,2}e_2,...,f_{n,1}e_1+f_{n,2}e_2)\|^{[q]}_{\mathbb{D},n}$ for $f_{1,1}e_1+f_{1,2}e_2,...,f_{n,1}e_1+f_{n,2}e_2\in L^p(\Omega_{\mathbb{D}})$

Let $fe_1+fe_2=|f_{1,1}e_1+f_{1,2}e_2|_k\vee ...\vee|f_{n,1}e_1+f_{n,2}e_2|_k$. Then for the case $p=q$,we define

$$\|(f_{1,1}e_1+f_{1,2}e_2,...,f_{n,1}e_1+f_{n,2}e_2)\|^{[p]}_{\mathbb{D},n} = \|fe_1+fe_2\|_{\mathbb{D}}$$
$$= \left(\int_{\Omega_{\mathbb{D}}}(|f_{1,1}e_1+f_{1,2}e_2|_k\vee ...\vee|f_{n,1}e_1+f_{n,2}e_2|_k)^p\right)^{\frac{1}{p}} \quad (1.12)$$

. Next,let X be the Banach lattice on $L^p(\Omega_{\mathbb{D}})$. Then using equation (1.12) the corresponding $\mathbb{D}$-valued lattice multi-norm $\left\{(X^n,\|.\|_{\mathbb{D},n} : n\in\mathbb{N})\right\}$ is obtained as

$$\|(f_{1,1}e_1+f_{1,2}e_2,...,f_{n,1}e_1+f_{n,2}e_2)\|^{L}_{\mathbb{D},n} = \left(\int_{\Omega_{\mathbb{D}}}(|f_{1,1}e_1+f_{1,2}e_2|_k\vee ...\vee|f_{n,1}e_1+f_{n,2}e_2|_k)^p\right)^{\frac{1}{p}}$$
$$= \|(f_{1,1}e_1+f_{1,2}e_2,...,f_{n,1}e_1+f_{n,2}e_2)\|^{[p]}_{\mathbb{D},n}.$$

Thus the lattice hyperbolic multi-norm and the standard p-hyperbolic multi-norm based on X coincide.

For dual of p , which is denoted by t i.e., $t=p^{\star}$ and also for $u_{1,1}e_1+u_{1,2}e_2,...,u_{n,1}e_1+u_{n,2}e_2 \in L^t(\Omega_{\mathbb{D}})$ and $n\in\mathbb{N}$ the dual of the standard p - hyperbolic valued multi-norm based on $L^p(\Omega_{\mathbb{D}})$ is given by

$$\|(u_{1,1}e_1+u_{1,2}e_2,...,u_{n,1}e_1+u_{n,2}e_2)\|^{[t]}_{\mathbb{D},n} = \||u_{1,1}e_1+u_{1,2}e_2|_k+....+|u_{n,1}e_1+u_{n,2}e_2|_k\|_{L^t(\Omega_{\mathbb{D}})}.$$

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